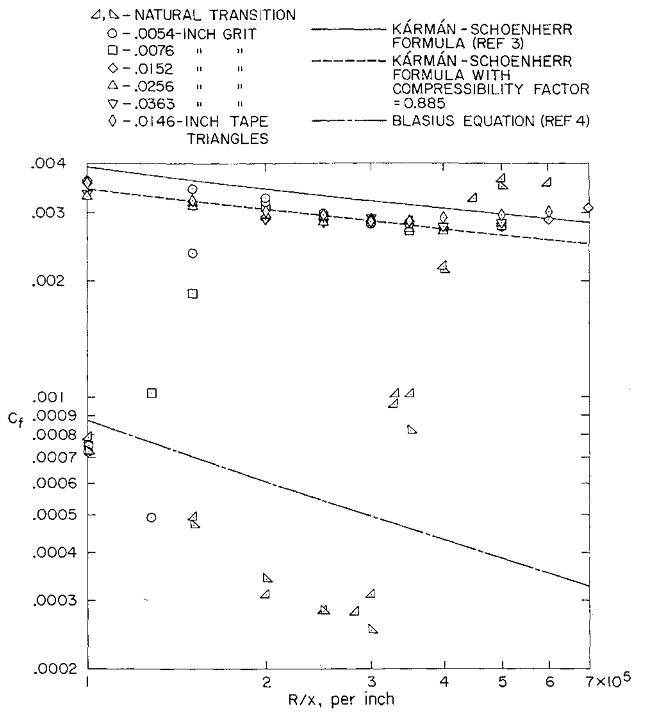

Fig. 1 Local skin friction (measured 5.875 in. behind the leading edge of plate) for various unit Reynolds numbers.

friction obtained from the incompressible Kármán-Schoenherr formula,[3] and from the formula modified by a compressibility factor of 0.885, together with laminar flow values obtained from the Blasius equation.[4]

It is interesting to note the consistency and character of these data. The data indicate that natural boundary-layer transition from laminar to turbulent flow occurred behind the floating element for $R/x \leq 3 \times 10^5$. The discrepancies between these laminar data and the Blasius curve are probably associated with the fact that, for the laminar data, the output signal from the friction balance was less than 8% of the manufacturer's full-scale rating; balance selection for these tests was based on the necessity of measuring the higher skin-friction values associated with all-turbulent flow. The "overshoot" of c_f at $R/x \geq 5 \times 10^5$ shows that natural transition occurred a short distance forward of the floating element. Similarly, the data indicate that the smaller sized (0.0054 and 0.0076 in.) grit-type trips were effective in producing all-turbulent flow only if $R/x > 2 \times 10^5$. However, the tape triangles and the larger sized (0.0152 in. and larger) grit transition trips produced essentially the same c_f for any particular test Reynolds number. For $R/x \leq 4 \times 10^5$ the results from these larger sized trips agree well with the Kármán-Schoenherr formula modified by the 0.885 compressibility factor. For $R/x > 4 \times 10^5$ the measurements downstream of the trips indicate friction coefficients that are not only greater than the compressible Kármán-Schoenherr values but even larger than the incompressible values. Recalibration of the balance at the conclusion of the tests failed to reveal any nonlinearities which would explain these results at the higher Reynolds numbers.

Several conclusions as to trip effectiveness and influence on local skin-friction coefficient can be drawn regarding the ratio of the grit size, k, to the boundary-layer thickness, δ (evaluated at the trip position by the method of Ref. 5). First, it can be concluded that the grit size has to be approximately the thickness of the boundary layer in order to be effective as a trip in producing all-turbulent skin friction. At $R/x = 2 \times 10^5$, k/δ is 0.8 and 1.1 for $k = 0.0054$ and 0.0076, respectively. Second, there is no apparent indication of a limitation on the k/δ ratio once all-turbulent skin-friction levels are developed at the floating element. In this study at $R/x = 4 \times 10^5$ the measured skin-friction coefficients are essentially the same although k/δ varied from 1.1 to 7.5. Third, the relationship between the distortion of the boundary layer due to the use of large-size grit-type transition trips and the skin friction downstream of the trips is not clearly evident from these results since the transition-fixed values differed from the desired all-turbulent skin-friction values only at the higher Reynolds numbers. However, the lack of dependence of the measured skin friction upon the grit size used to induce transition lends encouragement to eventual verification of the test technique applied to drag evaluation studies—provided a clear-cut means can be determined to account for the pressure drag of the trip elements.

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High Mach Number Viscous Flow past a Cylinder

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Lighthill[1] demonstrated the existence of a similar solution, in the inviscid approximation, of the flow in the region between a spherical shock wave and the (corresponding) body when the upstream Mach number is infinite and (consequently) the variations of density in this region are negligible; the stream function in this case is the sum of powers of the radial distance. Later, Whitham[2] extended this study to the case of a cylindrical shock-wave and obtained the solution in terms of (modified) Bessel functions. For the case of the spherical shock wave, Oberai[3] incorporated the effects of the (viscous) boundary layer near the body surface by expanding various flow variables in powers of $\mathrm{Re}^{-1/2}$ ($= \epsilon$) and obtained terms up to the order ϵ^2 (as explained in the section entitled "fully viscous flow" of this Note, calculation of these terms necessitates correcting the Rankine-Hugoniot relations). In this Note are reported the corresponding results for the case of a cylindrical shock-wave.

Procedure and Results

Velocity (components) and density are nondimensionalized with respect to the upstream values and distances with respect to a, the nose radius of the corresponding body. The upstream Mach number is assumed to be very large ($M_\infty^{-2} \ll 1$), the density in the shock layer (the region between the shock

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wave and the body) is assumed to have the constant value α $(= (\gamma - 1)/(\gamma + 1))$ and the slip at the body surface is neglected. The parameter $\epsilon = (\mu_0/\rho_\infty U_\infty a)^{1/2}$ (where μ_0 is the value of the coefficient of viscosity in the shock layer) is moderately small.

In the Navier-Stokes equation written in the cylindrical coordinates, we introduce the stream function $\Psi(r,\theta)$, $[q_r = -1/r\ \partial\Psi/\partial\theta,\ q_\theta = \partial\Psi/\partial r]$. Assuming $\Psi(r,\theta)$ to be of the form $r\psi(r)\sin\theta$, limiting the attention to the vicinity of the stagnation region ($\sin\theta \sim \theta$, $\cos\theta \sim 1$) and eliminating pressure by cross differentiation, we obtain the following differential equation for ψ

$$(\psi\psi''' + 3\psi\psi''/r - \psi'\psi'' - 3\psi'^2/r - 3\psi\psi'/r^2) + \epsilon^2(\psi^{iv} + 6\psi'''/r + 3\psi''/r^2 - 3\psi'/r^3) = 0 \quad (1)$$

It is known that the shock-wave thickness is $O(\epsilon^2)$ while the boundary-layer thickness is $O(\epsilon)$. The region between the boundary layer and the shock wave is termed "the proper shock layer." In this layer we assume the following expansion for ψ

$$\psi(r;\epsilon) = \psi_0(r) + \epsilon\bar{\psi}_1(r) + \epsilon^2\psi_2(r) + \ldots$$

Similarly, in the boundary layer we write

$$\psi(r;\epsilon) = \bar{\psi}(\bar{r};\epsilon) = \epsilon\bar{\psi}_1(\bar{r}) + \epsilon^2\bar{\psi}_2(\bar{r}) + \ldots; \quad r = 1 + \epsilon\bar{r}$$

Substituting these expansions in (1) and separating coefficients of various powers of ϵ, we obtain the equations governing various order ψ's and $\bar{\psi}$'s.

Inviscid approximation

The solution for ψ_0 is obtained (by Whitham) subject to the Rankine-Hugoniot conditions at the outer edge (of the proper shock layer) and the condition $q_r(1) = 0$. The tangential momentum equation evaluated at the inner edge of the shock wave gives Δ_0, the shock stand-off distance to the zero order. The results are

$$\psi_0 = B_0 I_1(A_0 r)/r + C_0 K_1(A_0 r)/r$$

and

$$\Delta_0(\alpha = \tfrac{1}{6},\ \text{i.e., } \gamma = \tfrac{7}{5}) = 0.260$$

with

$$A_0 = [1 - \alpha]/[\alpha(1 + \Delta_0)]$$

$$B_0 = (1 + \Delta_0)\{K_1[(1 - \alpha)/\alpha] - (1 - \alpha)K_1'[(1 - \alpha)/\alpha]\}$$

and

$$C_0 = -(1 + \Delta_0)\{I_1(1 - \alpha)/\alpha - (1 - \alpha)I_1'[(1 - \alpha)/\alpha]\} \quad (2)$$

Here, I_1 and K_1 are the modified Bessel functions of the first and the second kind of order 1.

First-order boundary layer

The governing equation is

$$(d/d\bar{r})\ [\bar{\psi}_1''' + \bar{\psi}_1\bar{\psi}_1'' - (\bar{\psi}_1')^2] = 0$$

The known boundary conditions are $\bar{\psi}_1(0) = 0 = \bar{\psi}_1'(0)$; the remaining boundary conditions are obtained by matching with ψ_0, i.e.,

$$\text{as } \bar{r} \to \infty,\ \bar{\psi}_1 \sim k_1\bar{r} + \psi_1(1)$$

$$\text{or } \bar{\psi}_1' = k_1 \text{ where } k_1 = \psi_0'(1)$$

On setting $\bar{\psi}_1(\bar{r}) = k_1^{1/2}f_1(\eta)$ where $\eta = (k_1)^{1/2}\bar{r}$, f_1 is found to satisfy the Falkner-Skan equation; thus, the behavior of $\bar{\psi}_1$ for large $\bar{r}$ is given by

$$\bar{\psi}_1 \infty k_1\bar{r} - (k_1)^{1/2}\beta_1 + \exp$$

where $\beta_1 = 0.6479$ and exp denotes exponentially small terms.

Displacement flow

To the order ϵ the shock wave is still a thin discontinuity; therefore, the outer boundary conditions for ψ_1 are the Rankine-Hugoniot relations applied at $r = 1 + \Delta_0 + \epsilon\Delta_1$ where $\epsilon\Delta_1$ is the change in the shock stand-off distance due to the (first-order) boundary layer. The inner boundary condition, obtained by matching with $\bar{\psi}_1$, is $\psi_1(1) = -\beta_1(k_1)^{1/2}$. The solution is

$$\psi_1 = -[\Delta_1/(1 + \Delta_0)]\psi_0' \quad (3)$$

$$\Delta_1 = (1 + \Delta_0)\beta_1(k_1)^{-1/2}$$

Second-order boundary layer

By setting $\bar{\psi}_2(\bar{r}) = f_2(\eta)$, f_2 is found to satisfy the following differential equation

$$(d/d\eta)(f_2''' + f_1f_2'' - 2f_1'f_2' + f_1''f_2) = 3(1 - f_1''') \quad (4)$$

Again, we know that $f_2(0) = 0 = f_2'(0)$, while the matching gives:

$$\text{as } \eta \to \infty,\ f_2 \sim k_2\eta^2/2 + l_2\eta + \psi_2(1)$$

$$\text{or } f_2' \sim k_2\eta + l_2$$

where $k_2 = \psi_0''(1)/k_1$ $(= 3)$ and $l_2 = k_1^{-1/2}\psi_1'(1)$. Equation (4) can be integrated once to obtain

$$f_2''' + f_1f_2'' - 2f_1'f_2' + f_1''f_2 = -3f_1'' + k_2(\beta_1 + \eta) - 2l_2$$

Following Van Dyke,[4] f_2 may be split up as $f_2 = f_{2c} + f_{2v} + f_{2d}$ and the linear third-order differential equations for f_{2c}, f_{2v} and f_{2d} solved separately. In particular, the behavior near infinity is found to be

$$\text{as } \bar{r} \to \infty,\ \bar{\psi}_2 \sim \tfrac{1}{2}k_1k_2\bar{r}^2 + k_1^{1/2}l_2\bar{r} - \beta_2$$

where

$$\beta_2 = -0.3239l_2 - 0.4128k_2 - 0.7133$$

Fully viscous flow

As regards the differential equation for ψ_2 and the corresponding boundary conditions there are two important observations to be made. 1) ψ_2 will satisfy a third-order nonhomogeneous equation. As the highest derivative, ψ_2''', has the coefficient ψ_0 the equation would have been singular along $r = 1$ (where $\psi_0 = 0$) but for the fact that in view of the solutions (2) and (3) the nonhomogeneous part is also found to have a common factor ψ_0 (2). The inner boundary condition obtained by matching is $\psi_2(1) = -\beta_2$. As regards the boundary conditions at the outer edge, we have to use the modified shock-wave conditions.[5] As explained in Ref. 3, these conditions involve an arbitrary constant L which is the legacy of the solution of the zero-order shock-wave structure. In the same way as ψ_0 and ψ_1 determine, respectively, the shock stand-off distance Δ_0 and the correction $\epsilon\Delta_1$, ψ_2 will determine L where ϵ^2L is the combined effect of the second-order boundary layer and the shock-wave curvature. Also, as we are dealing with quantities of order ϵ^2 and the shock-wave thickness itself is $0(\epsilon^2)$ we can no longer talk of the shock stand-off distance, but of the distance between the body and a particular point in the shock wave; e.g., the point where a specified value of the normal velocity in the zero-order shock-wave structure is attained.

Adopting a procedure similar to that of Ref. 3, we find that the distance between the body and the point where the normal velocity in the zero-order shock-wave structure (solved by assuming $Pr = \frac{3}{4}$, $\mu \propto T^{1/2}$, perfect gas with constant specific heats and Stokes' relation between the viscosity coefficients) is sonic and is given by

$$\Delta(\alpha = \tfrac{1}{6}) = 0.260 + 0.9379\epsilon + 2.884\epsilon^2 + \ldots.$$

Finally, it may be noted that 1) for the effect of the corrections in the Rankine-Hugoniot relations on the surface conditions one has to calculate the third-order boundary layer and 2) following Ref. 3 the analysis can be extended to the case when the flow in the shock layer (including the boundary layer) is considered compressible.

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Derivatives of Eigenvalues and Eigenvectors

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Introduction

MANY times it is desirable or even mandatory to find the effects of a design parameter change on the dynamic stability and/or response characteristics of a system. This occurs, for example, when a design is unsatisfactory and improvement is sought on a cut and dry basis, sometimes guided by the most flimsy (but only available) logic. Effects predicted by incrementing the parameter and finding the new solution can be sensitive to inherent numerical difficulties as well as costly. Costs include use of highly skilled personnel when there are many severe demands for their time.

It is highly desirable to have available an accurate, efficient tool to compute directly and efficiently the effects of a design parameter change. Such a tool would receive wide application in finding the gradient of a dynamic type constraint variable as a function of the design parameters in an automated optimization procedure. It would also be valuable in less sophisticated design studies. It has the highly significant advantage of contributing to the physical understanding and insight of a problem.

The homogeneous part of a set of arbitrary order, linear, constant coefficient differential equations may be written in first-order matrix form

$$A\dot{y} + By = 0 \tag{1}$$

where the $N \times N$ matrices A and B are functions of the design parameters, and need not be symmetric, real, or hermitian. Previously published results are restricted to the symmetric case. Some of the y coordinates are the original displacement coordinates, some may be first derivatives of the original displacement coordinates, etc. Assuming the solution

$$y = \varphi e^{\alpha t} \tag{2}$$

leads to the eigen-problem

$$\alpha_i A\varphi_i + B\varphi_i = 0 \tag{3}$$

which has the characteristic equation

$$|\alpha A + B| = 0 \tag{4}$$

where α is an eigenvalue and φ is the corresponding eigenvector.

Consider the eigen-problem

$$\beta_j A^T\theta_j + B^T\theta_j = 0 \tag{5}$$

which has the characteristic equation

$$|\beta A^T + B^T| = |\beta A + B| = 0 \tag{6}$$

the same as (4). Equation (5) becomes

$$\alpha_j A^T\theta_j + B^T\theta_j = 0 \tag{7}$$

Premultiplying (3) by $\theta_j{}^T$ and (7) by $\varphi_i{}^T$, transposing the latter, and subtracting gives

$$(\alpha_i - \alpha_j)\theta_j{}^T A\varphi_i = 0 \tag{8}$$

The following orthogonality relations result:

$$\theta_j{}^T A\varphi_i = \theta_j{}^T B\varphi_i = 0,\ i \neq j \tag{9}$$

assuming distinct eigenvalues.

Apparently Traill-Nash[1] was the first to develop orthogonality relations for nonsymmetric matrices and apply them to dynamics. See also Halfman[2] and Foss.[3]

Since there are two sets of eigenvectors involved here, two normalization conditions must be imposed. It is convenient to normalize φ_i such that the element corresponding to a displacement coordinate with the largest modulus is set equal to unity, then to normalize θ_i such that

$$\theta_i{}^T A\varphi_i = 1 \tag{10}$$

Note that it is neither necessary to solve both the eigenproblems, nor to find all N solutions, because

$$\Theta_{n\times N}{}^T A_{N\times N}\Phi_{N\times n} = I_{n\times n},\ n \leq N$$

and

$$\Theta^T = \Phi^T[A\Phi\Phi^T]^{-1} \tag{11}$$

where Θ and Φ are matrices whose columns are the eigenvectors θ and φ, respectively.

Derivative of an Eigenvalue

Now start with the Rayleigh Quotient written as

$$\alpha_i\theta_i{}^T A\varphi_i + \theta_i{}^T B\varphi_i = 0 \tag{12}$$

and take the partial derivative with respect to a parameter ρ_k

$$\alpha_{i,k}\theta_i{}^T A\varphi_i + \alpha_i\theta_{i,k}{}^T A\varphi_i + \alpha_i\theta_i{}^T A_{,k}\varphi_i + \alpha_i\theta_i{}^T A\varphi_{i,k} + \theta_{i,k}{}^T B\varphi_i + \theta_i{}^T B_{,k}\varphi_i + \theta_i{}^T B\varphi_{i,k} = 0 \tag{13}$$

Collecting terms gives

$$\alpha_{i,k}\theta_i{}^T A\varphi_i + \theta_{i,k}{}^T\{\alpha_i A\varphi_i + B\varphi_i\} + \theta_i{}^T[\alpha_i A_{,k} + B_{,k}]\varphi_i + \lfloor\alpha_i\theta_i{}^T A + \theta_i{}^T B\rfloor\varphi_{i,k} = 0 \tag{14}$$

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